Estimating Supermodular Games using Rationalizable Strategies*

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Abstract

We propose a set-estimation approach to supermodular games using the restrictons of rationalizable strategies, which is a weaker solution concept than Nash equilibrium. The set of rationalizable strategies of a supermodular game forms a complete lattice, and are bounded above and below by two extremal Nash equilibria. We use a well-known alogrithm to compute the two extremal equilibria, and then construct moment inequalities for set estimation of the supermodular game. Finally, we conduct Monte Carlo experiments to illustrate how the estimated confidence sets vary in response to changes in the data generating process.

Keywords: Supermodular games, Rationalizability, Moment inequalities
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1 Introduction

Recently, a number of studies have estimated supermodular games (e.g., Jia, 2008, Ackerberg and Gowrisankaran, 2006, Matvos and Ostrovsky, 2010, and Nishida, 2012). One of the main issues on estimating a supermodular game is how to address (potential) multiplicity of equilibria. Supermodular games may have multiple equilibria, and identifying which equilibrium is played in the data is not straightforward in general. In such a case, some sort of equilibrium selection rule is imposed a priori in most cases.

In this note, we propose a way to estimate supermodular games without imposing any equilibrium selection rule. In particular, we relax the Nash equilibrium assumption that the researchers typically impose in order to estimate game-theoretic models, and use rationalizability as a solution concept in our estimation.[1] More precisely, we exploit the lattice property of the set of rationalizable strategies of supermodular games, and then apply a moment inequality estimator. The set of rationalizable strategies in supermodular games has a lattice structure, which implies that the set of rationalizable strategies is bounded above and below by the strategies of the two extremal Nash equilibria, say $\overline{s}$ and $\underline{s}$. These two extremal Nash equilibrium strategies bound any rationalizable strategies s^* for each player, i.e., $s_i^* \geq \underline{s}_i$ and $\overline{s}_i \geq s_i^*$ for all i, where $\geq$ denotes appropriately specified partial order over the set of strategies.

Furthermore, we utilize the well-known result by Milgrom and Roberts (1990): The two extremal Nash equilibrium strategies $\overline{s}_i$ and $\underline{s}_i$ can be found by applying the best response correspondences iteratively starting from the largest and the smallest elements in the set of strategies (which monotonically converges to $\overline{s}$ and $\underline{s}$, respectively). Thus, we can easily compute $\overline{s}$ and $\underline{s}$, which allows us to construct moment inequalities based on $s_i^* \geq \underline{s}_i$ and $\overline{s}_i \geq s_i^*$.[2]

After proposing our estimation strategy, we present the results of Monte Carlo experiments to show how our estimation strategy works using a simple

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[1] Rationalizability (Bernheim, 1984, and Pearce, 1984) is a weaker concept than Nash equilibrium. Hence, all Nash equilibria are rationalizable, while rationalizability does not imply that the strategy profile constitutes a Nash equilibirum.

[2] The appraoch we propose is similar to Uetake and Watanabe (2012b), which estimate a two-sided matching model to study banks' entry and merger decisions. They use the property that the set of stable matchings in two-sided matching models can be characterized by Tarski's fixed point theorem in the similar way as the set of equilibria of supermodular games. Uetake and Watanabe (2012a) propose another way to exploit the lattice property of the stable matching to estimate two-sided matching models with non-transferable utilities.

investment game with complementarity. We find that the obtained confidence set includes the true parameter value. Also, we illustrate how confidence sets vary with changes in the data generating process.

This note relates to a few strands of the literature on estimation of games. First, this note adds to the literature on estimation of games without equilibrium assumptions such as Nash equilibrium. In particular, it is related to Aradillas-Lopez and Tamer (2008), which considers the identification power of rationalizability as well as level-k rationality in comparison with Nash equilibrium. This note differs from theirs by considering supermodular games specifically. Because we focus on supermodular games, we can exploit the theoretical properties of rationalizable strategies in our estimation. In their comment to Aradillas-Lopez and Tamer (2008), Molinari and Rosen (2008) proposes a similar approach to ours for level-k rationality for differentiated product pricing game, which is a specific examples of supermodular games. This note differs from theirs as our focus is on rationalizability instead of level-k rationality. Also, we add to their approah by using monotone comparative statics results in constructing moment inequalities based on utilitities.

The second related literature is the literature that studies identification and estimation of models using monotone comparative statics. Echenique and Komunjer (2009) proposes a test on complementarities using monotone comparative statics property when the model may have multiple equilibria. Lazzati (2012) also uses monotone comparative statics to study partial identification of treatment response models with endogenous social interactions. Third, the note more broadly relates to studies addressing the issue of multiple equilibria in games using a moment inequality estimator (e.g., Ciliberto and Tamer, 2009, Ho, 2009, Kawai and Watanabe, 2013).

We define supermodular games and summarize several important results of supermodular games in the next section. In Section 3, we discuss our strategy to estimate supermodular games. We present our Monte Carlo experiments in Section 4, and the conclusion follows in Section 5.

2 Supermodular Games

Consider a n-player normal form game $G = ((I, \{S_i, \succeq_i\}_{i \in I}, \{u_i\}_{i \in I}))$. We denote the set of players by I, i.e., $i \in I = \{1, 2, ..., n\}$. Each player i's strategy space $(S_i, \succeq_i)$ is a complete lattice. Let $S = \prod_{i=1}^{n} S_i$. Each player i's utility function $u_i : S \to \mathbb{R}$ is order upper-semicontinuous (see Milgrom and

Roberts, 1990, for definition). Now, we define a supermodular game.

Definition 1 *A normal form game* $G = (I, \{S_i, \succeq_i\}_{i \in I}, \{u_i\}_{i \in I})$ *is a supermodular game if*

1. u_i *is supermodular in* S_i*, i.e., for all* $s_i, s_i' \in S_i$*, and for all* $s_{-i} \in S_{-i}$*,*

$$u_i(s_i \wedge s_i', s_{-i}) + u_i(s_i \vee s_i', s_{-i}) \geq u_i(s_i, s_{-i}) + u_i(s_i', s_{-i}),$$

and

2. u_i *has increasing difference in* S_i *and* S_{-i}*, i.e., for all* $s_i, s_i' \in S_i$ *and* $s_{-i}, s_{-i}' \in S_{-i}$ *such that* $s_i \succeq_i s_i'$ *and* $s_{-i} \succeq_{-i} s_{-i}'$*,*

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}') - u_i(s_i', s_{-i}')$$

The following example is a complete information chain-store entry game studied by Jia (2008).

Example 1 (Jia (2008)) *Consider an entry game by two chain stores, Walmart and Kmart, i.e.,* $I = \{\textit{Walmart, Kmart}\}$*. Each chain store's entry decision in market* m *is denoted by* $s_{im} \in \{0, 1\}$*, where* 0 *means stay out and* 1 *entry. Then, player* i*'s strategy space is* $S_i = \{0, 1\}^M$ *and* $s_i \succeq_i s_i'$ *if and only if* $s_{im} \geq s_{im}'$ *for all* $m = 1, 2, ..., M$*, where* M *is the number of markets. The (simplified) utility function Jia uses is as follows.*

$$u_i(s_i, s_j) = \sum_{m=1}^{M} \left[s_{im} \times \left(\delta_{ii} \sum_{l \neq m} \left(\frac{s_{il}}{Z_{ml}} \right) + \delta_{ij} s_{jm} + \varepsilon_{im} \right) \right],$$

where Z_{ml} *is the distance between market* m *and* l*,* δ_{ii} *is the positive spillover effect of firm* i*'s entry in market* l *on firm* i*'s profit of market* m*,* δ_{ij} *is the business stealing effect by the existence of firm* j*, and* ε_{im} *is shock on profits, which is not observed by an econometrician. She shows that the chain-store game with two players is a supermodular game if* $\delta_{ii} > 0$*.*

Other empirical applications of supermodular games include a technology adoption game with network effects in the banking industries studied by Ackerberg and Gowrisankaran (2006) and mutual funds' proxy voting decisions with peer effects as in Matvos and Ostrovsky (2010). Also, Nishida (2012) extends Jia's analysis by incorporating multiple branching decisions in the Japanese convenience store industry.

In supermodular games, the existence of Nash equilibrium and its characterization are given by Tarski's fixed point theorem (Tarski, 1955) and Topkis's monotonicity theorem (Topkis, 1968, 1998).[3] To apply Tarski's fixed point theorem to supermodular game G, we first note that $s^* \in S$ is a Nash equilibrium if and only if the best-response correspondence, $BR_i(s) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$, satisfies $s_i^* \in BR_i(s^*)$ for all $i \in I$. In other words, the set of Nash equilibria is the set of fixed points of best response correspondences $BR : S \rightrightarrows S$, where $BR = \{BR_i\}_{i \in I}$. Moreover, for supermodular games, it is known that the best-response correspondence is non-decreasing.[4]

Now, we summarize important results by Milgrom and Roberts (1990), which we will use for estimation. Applying Tarski's fixed point theorem and Topkis's monotonicity theorem, Milgrom and Roberts (1990) give useful characterizations of the set of Nash equilibria and rationalizable strategies of supermodular games. The first two results concern characterizations of the set of Nash equilibria, while the third result discusses characterization of rationalizable strategies, on which our estimation strategy is relied.

Theorem 1 (Milgrom and Roberts, 1990) *The set of Nash equilibria of a supermodular game is a complete lattice. Hence, it has a largest and smallest element.*

Moreover, we can compute the greatest and smallest element of the set of Nash equilibria using the following iterative best response algorithm.

Corollary 1 (Milgrom and Roberts, 1990) *There exist the largest and smallest element in the set of Nash equilibria, $\overline{s}^*$ and $\underline{s}^*$. Moreover, these extremal equilibria are achieved by applying $BR : S \rightrightarrows S$ recursively starting from $\underline{s} = \inf\ S$ and $\overline{s} = \sup\ S$ respectively.*

Furthermore, it is known that the two extremal Nash equilibria, $\overline{s}^*$ and $\underline{s}^*$, in fact provide lower and upper bounds for the set of rationalizable strategies.[5]

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[3] Formally, Tarski's fixed point theorem is as follows: If a set T is a complete lattice and $f : T \to T$ is a non-decreasing function, then f has a fixed point. Moreover, the set of fixed points has its largest ad smallest element in T. Moreover, Topkis's Monotonicity Theorem is as follows: Let X be a complete lattice and T a partially ordered set. Suppose $F : X \times T \to R$ has increasing differences in $(x, t) \in X \times T$ and is supermodular in $x \in X$. Then $\arg\max_{x \in X} F(x, t)$ is monotone non-decreasing in (x, t).

[4] Formally, we can use a Topkis's (1968) result in order to prove that the best response correspondence is non-decreasing.

[5] For the formal definition of rationalibale strategy, see, e.g., Bernheim (1984) or Pearce (1984).

Theorem 2 (Milgrom and Roberts, 1990) *The set of rationalizable strategies of a supermodular game has largest and smallest elements. Moreover, those extremal strategy profiles correspond to extremal Nash equilibrium strategy profiles $\overline{s}^*$ and $\underline{s}^*$.*

In the estimation section below, we use these results to construct moment inequalities.

3 Estimation

In this section, we propose an estimation strategy for supermodular games based on moment inequalities. Our estimation strategy exploits the lattice structure of the set of rationalizable strategies which allows partial ordering over the set of strategies. We use this property to construct inequalities and apply a moment inequalities estimator.

We provide two ways to construct moment inequalities in this section. The first approach is to construct moment inequalities in the strategy space, S, and the second approach is to do so in the utility space, $\mathbb{R}$. In both of the cases, we use the property that strategies or utilities corresponding to the two extremal equilibria bound (above and below) all rationalizable strategies or corresponding utilities.

3.1 Moment Inequalities Based on Strategy

Consider a supermodular game $G = (I, \{S_i, \succeq_i\}_{i \in I}, \{u_i\}_{i \in I})$. We specify the player i's utility function as $u_i(s) = f(s, \mathbf{x}_i, \mathbf{z}; \theta) + \varepsilon_{is_i}$, where $f(\cdot)$ is the deterministic part of the utility, $\mathbf{x}_i$ is the vector of each player i's characteristics and $\mathbf{z}$ is the vector of exogenous market-level characteristics. The error term ε_{is} captures random payoff shock drawn from distribution g, which is observed by the players but unobserved by the econometrician. We parameterize the utility function and the distribution of the taste shock g by parameter $\theta \in \Theta$. The data environment we consider is the case in which the econometrician observes the game to be played independently across M markets, indexed by $m \in \{1, ..., M\}$, where M is large. Note also that we consider that observations $(I_m, s_m^{DATA}, \mathbf{x}_m, \mathbf{z}_m)$, $m = 1, 2, ..., M$, are realized as one of the rationalizable strategies conditional on $\mathbf{x}_m$ and $\mathbf{z}_m$, i.e., we do *not* assume that the data is a realization of a Nash equilibrium.

Let S^* be the set of all Nash equilibria, i.e., $S^* = \{s \in S : s^* \in BR(s^*)\}$, and denote the two extremal Nash equilibria as $\overline{s}^* = \sup\ S^*$ and $\underline{s}^* = \inf\ S^*$. Note that the extremal equilibrium is a function of all players' characteristics, $\mathbf{x} = \{\mathbf{x}_i\}_{i\in I}$, market characteristics, $\mathbf{z}$, and the parameter θ. The researcher cannot identify which rationalizable strategy corresponds to the observed data. However, the observed data s^{DATA}, which correspond to one of the rationalizable outcomes, is ordered between $\overline{s}^*$ and $\underline{s}^*$ as evident from the fact that the set of rationalizable strategies of supermodular games has the lattice structure, and bounded by $\overline{s}^*$ and $\underline{s}^*$ as in Theorem 2. Hence, we obtain the following relationships given a set of shocks: for all $i \in I$,

$$\begin{aligned} \overline{s}_i^* &\succeq_i s_i^{DATA}, & (1)\\ s_i^{DATA} &\succeq_i \underline{s}_i^*. & (2) \end{aligned}$$

This is the basis of our estimation strategy using moment inequalities.

The relationship above cannot be directly used in the estimation, because the inequality relationships, $\succeq_i$, is not defined in terms of real values, but in terms of the partial order on S_i. In most applications, however, we can find a way to transform the space of S_i such that we can define some distance between s_i and $s_i' \in S_i$ without losing the partial ordering $\succeq_i$. For notational simplicity, we consider the case that S_i is simply $\mathbb{R}$ in the following. Then, we can construct the moment inequalities as

$$\begin{aligned} E\left[\sum_{i\in I}\left(\overline{s}_i^*(\theta,\mathbf{x},\mathbf{z}) - s_i^{DATA}\right)\middle|\,\mathbf{x},\mathbf{z}\right] &\geq 0, & (3)\\ E\left[\sum_{i\in I}\left(s_i^{DATA} - \underline{s}_i^*(\theta,\mathbf{x},\mathbf{z})\right)\middle|\,\mathbf{x},\mathbf{z}\right] &\geq 0, & (4) \end{aligned}$$

and the corresponding sample analogues are written as

$$\begin{aligned} \frac{1}{M}\sum_{m\in M}\sum_{i\in I_m}\left(\overline{s}_{i,m}^*(\theta,\mathbf{x}_m,\mathbf{z}_m) - s_{i,m}^{DATA}\right)\times g(\mathbf{x}_m,\mathbf{z}_m) &\geq 0,\\ \frac{1}{M}\sum_{m\in M}\sum_{i\in I_m}\left(s_{i,m}^{DATA} - \underline{s}_{i,m}^*(\theta,\mathbf{x}_m,\mathbf{z}_m)\right)\times g(\mathbf{x}_m,\mathbf{z}_m) &\geq 0, \end{aligned}$$

where $g(\cdot)$ is any non-negative valued function of firm and market characteristics.

3.2 Moment Inequalities Based on Utility

In some cases, constructing moment inequalities based on utility (instead of strategy) can be easier because the mapping between strategy and utility may not be very straightforward. In such a case, one can also construct moment inequalities in terms of utility associated with rationalizable strategies. First, we describe the cases in which such construction is possible. To do so, we first define *positive spillover* property as follows.

Definition 2 *Utility function* $u_i(s)$ *satisfies positive spillover property if* $u_i(s_i, s_{-i}) \geq u_i(s_i, s'_{-i})$ *for all* i *whenever* $s_{-i} \succeq_{-i} s'_{-i}$.

This property means that the degree of complementarity increases as "greater" strategy is chosen by other players. Milgrom and Roberts (1990) show monotone comparative statistics results for supermodular games with positive spillover.

Theorem 3 (Milgrom and Roberts (1990)) *Suppose* $G_\theta = ((I, \{S_i, \succeq_i\}_{i\in I}, \{u_i\}_{i\in I}), \theta)$ *is a supermodular game with positive spillovers. Then the rationalizable strategies are ordered in accordance with Pareto preference, i.e., for any rationalizable strategies* s^*,

$$u_i(\overline{s}^*(\theta)) \geq u_i(s^*;\theta) \geq u_i(\underline{s}^*(\theta)),\ \forall i \in I \tag{5}$$

Theorem 4 does not necessarily imply the largest Nash equilibrium $\overline{s}^*$ is Pareto optimal. This theorem shows that the largest Nash equilibrium $\overline{s}^*$ is most preferred to any rationalizable strategies for any player, and the smallest Nash equilibrium $\underline{s}^*$ is least preferred to any other rationalizable strategies for any player. As in equations (3) and (4), observed strategies provide utilities between the largest and smallest Nash equilibria, and we can construct moment inequalities as follows;

$$\begin{aligned}
E\left[\sum_{i\in I} u_i(\overline{s}^*(\theta, \mathbf{x}, \mathbf{z})) - u_i(s^{DATA};\theta) \big| \, \mathbf{x}, \mathbf{z}\right] &\geq 0, \qquad (6)\\
E\left[\sum_{i\in I} u_i(s^{DATA};\theta) - u_i(\underline{s}^*(\theta, \mathbf{x}, \mathbf{z})) \big| \, \mathbf{x}, \mathbf{z}\right] &\geq 0. \qquad (7)
\end{aligned}$$

The corresponding sample analogues of moment inequalities are written as

$$\begin{aligned}
\frac{1}{M}\sum_{m\in M}\sum_{i\in I_m} \big(u_{i,m}(\overline{s}^*(\theta, \mathbf{x}_m, \mathbf{z}_m)) - u_{i,m}(s^{DATA};\theta)\big) \times g(\mathbf{x}_m, \mathbf{z}_m) &\geq 0,\\
\frac{1}{M}\sum_{m\in M}\sum_{i\in I_m} \big(u_{i,m}(s^{DATA};\theta) - u_{i,m}(\underline{s}^*(\theta, \mathbf{x}_m, \mathbf{z}_m))\big) \times g(\mathbf{x}_m, \mathbf{z}_m) &\geq 0.
\end{aligned}$$

Comments on Estimation A few comments are in order. First, our estimation strategy is computationally simple and do not assume any equilibrium selection mechanism. In fact we use rationalizability as a solution concept, which is a much weaker concept than Nash equilibrium, and do not assume which rationalizable strategy is realized in the data *a priori*. Since any rationalizable strategy, including one observed in the data, is bounded by $\overline{s}^*$ and $\underline{s}^*$, we do not need any assumption about selection mechanism. Another approach, for example, would be to use an equilibrium assumption and apply the estimation strategy by Bajari *et al.* (2010), in which one can use Echenique (2007) that provides an efficient algorithm to compute all Nash equilibria of a supermodular game.

Second, the identified set defined by these moment inequalities is not sharp in general. Berry and Tamer (2006) define the sharp identified set as the set of parameters θ that are consistent with the data and the model. Heuristically, we say θ is in the identified set if and only if there exists a (proper) equilibrium selection mechanism such that the induced probability distribution of outcome of the game matches the choice probabilities observed in the data almost everywhere. If Nash equilibrium were used as our solution concept, our identified set would not be sharp, because it might include infeasible parameters θ for which it is not possible to find any equilibrium selection mechanism. The identified set is sharp in case if there are only two players or in case if we use correlated rationalizability as a solution concept. This is due to the fact that the set of serially undominated strategies coincides with the set of rationalizable strategies in these two cases. Since we use rationalizability as our solution concept, however, the identified set is not necessarily sharp.

Estimation Algorithm Let us denote the moment inequalities by $E\left[h(\mathbf{x}, \mathbf{z}; \theta)\right] \geq 0$. Our inference is based on observations from many markets indexed by $m = 1, 2, ..., M$. The estimation procedure is as follows.

1. Fix parameter θ. For each market $m = 1, ..., M$, draw large number of $\boldsymbol{\varepsilon}^{ms} = \{\varepsilon_{is_i}^{ms}\}_{i=1}^{I}$ from g, where the number of simulation draws is S, and s denotes s-th simulation draw.

2. For each draw $\boldsymbol{\varepsilon}^{ms}$ in each market m, compute $\overline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms})$ and $\underline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms})$ by iteratively applying the best response correspondence for each player.

3. Construct sample analogue of moment inequalities using $\overline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms})$ and $\underline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms})$ (or $u_{i,m}(\overline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms}))$ and $u_{i,m}(\underline{s}^*(\theta, \mathbf{x}, \mathbf{z}, \boldsymbol{\varepsilon}^{ms})))$

as well as the observation on s^{DATA} (or $u_{i,m}(s^{DATA})$):

$$\frac{1}{MS}\sum_{m=1}^{M}\sum_{s=1}^{S}\sum_{i\in I_m} h(\mathbf{x}_m, \mathbf{z}_m, \boldsymbol{\varepsilon}^{ls}; \theta) \geq 0$$

4. Use moment inequalities estimator, such as Chernozhukov, Hong, and Tamer (2007), Andrews and Soares (2010) and Pakes, Porter, Ho, and Ishii (2011).

4 Monte Carlo Experiment

In this section, we present the results of Monte Carlo experiments. For these experiments, we consider a simple supermodular game; two-player investment game with complementarity. Player $i \in \{1, 2\}$ chooses whether to make an investment ($s_i = 1$) or not ($s_i = 0$).[6] The utility function of Player i is written as

$$u_i(s_i, s_{-i}) = \begin{cases} \theta_1 \sum_{i\in\{1,2\}} s_i - x_i^{\theta_2} + \varepsilon_i & \text{if } s_i = 1 \\ 0 & \text{if } s_i = 0, \end{cases}$$

where (θ_1, θ_2) are parameters to be estimated, x_i is Player i's characteristics that affects i's cost of investment, and ε_i is an idiosyncratic shock.[7] We can interpret θ_1 as a parameter measuring complementarity of investments, and θ_2 as a parameter capturing convexity of the investment cost (we assume $\theta_1 > 0$ and $\theta_2 > 1$). As is clear from the specification of the utility function, each player's investment has complementarity and the game is a supermodular game.

The best response function of Player i given the strategy of the other player is written as

$$\begin{aligned} BR_i(s_j &= 0) = \begin{cases} 1 & \text{if } \theta_1 - x_i^{\theta_2} > -\varepsilon_i \\ 0 & \text{otherwise} \end{cases} \\ BR_i(s_j &= 1) = \begin{cases} 1 & \text{if } 2\theta_1 - x_i^{\theta_2} > -\varepsilon_i \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, given θ_1, θ_2, and x_i, we can draw the Nash equilibrium outcomes corresponding to the realizations of $(\varepsilon_1, \varepsilon_2)$ as in Figure 1. Points A and B

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[6] We focus on pure strategies in our analysis.

[7] We assume that the probability ditribution of ε_i is continuous. Hence, the probability that two choices give exactly the same payoff is zero.

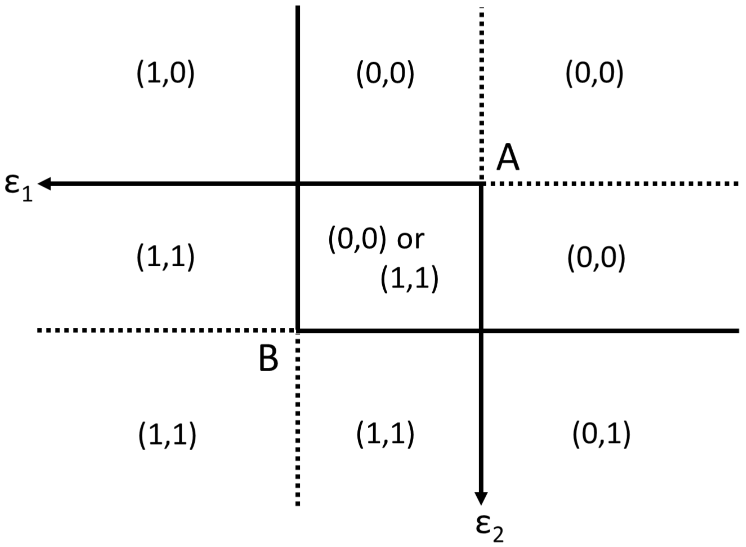

Figure 1: Numbers in each parenthesis corresponds to (s_1^*, s_2^*). Points A and B correspond to $(2\theta_1 - x_1^{\theta_2}, 2\theta_1 - x_2^{\theta_2})$ and $(\theta_1 - x_1^{\theta_2}, \theta_1 - x_2^{\theta_2})$, respectively. The area in the middle corresponds to the region with multiple equilibria.

in the Figure corresponds to $(2\theta_1 - x_1^{\theta_2}, 2\theta_1 - x_2^{\theta_2})$ and $(\theta_1 - x_1^{\theta_2}, \theta_1 - x_2^{\theta_2})$, respectively. The region in the middle of the Figure corresponds to the case that there are multiple equilibrium outcomes, while the rest of the regions have a unique equilibrium outcome. For example, very large values of ε_1 and ε_2 result in both players to invest (corresponding to the South-West corner of the Figure), while a large value of ε_1 and a small value of ε_2 result in Player 1 to invest and Player 2 not to invest (corresponding to the North-West corner of the Figure).

For the Monte Carlo experiments, we use the parameter values of $\theta_1 = 2$ and $\theta_2 = 2$. The number of markets is set at $M = 2000$. The value (x_1, x_2) of players' characteristics are uniformly distributed over the discrete values in the sets X_1 and X_2, respectively, where $X_1 = \{1.1, 1.2, 1.3, 1.4, 1.5\}$ and $X_2 = \{1.9, 2.8, 3.7, 4.6, 5.5\}$ in $m \in \{1, ..., 1000\}$, and $X_1 = \{1.9, 2.8, 3.7, 4.6, 5.5\}$ and $X_2 = \{1.1, 1.2, 1.3, 1.4, 1.5\}$ in $m \in \{1001, ..., 2000\}$. Finally, we specify that the error term ε_i follows a Normal distribution with mean 0 and standard error of 0.05.

In our experiment, we have three different data generating processes. In case if multiple equilibrium outcome is possible, we let the outcomes

DGP	Parameter	Confidence Set
$p = 0.5$		
	θ_1	$[1.00, 2.80]$
	θ_2	$[1.42, 3.08]$
$p = 0.1$		
	θ_1	$[1.02, 2.28]$
	θ_2	$[1.42, 2.20]$
$p = 0.9$		
	θ_1	$[1.02, 2.84]$
	θ_2	$[1.44, 3.14]$

Table 1: Minimum and maximum of the 95% confidence set for each parameter for different data generating processes.

$(0, 0)$ and $(1, 1)$ to occur with probabilities p and $1 - p$, and we vary the value of p. Specifically, we use three values $p = 0.1$, 0.5, and 0.9 in the experiment. Figure 2 is the plot of the confidence set in each case. In our implementation, we construct 95% confidence sets using Andrews and Soares (2010)'s Generalized Moment Selection.

Figure 2 presents the 95% confidence set for all three cases, and Table 1 presents the minimum and the maximum of the parameter values for each dimension of the confidence set. In all cases, the true parameter value of (θ_1,θ_2) is included in the confidence set regardless of the data generating process. Thus, the approach we propose works fine in our Monte Carlo experiments regardless of which equilibrium is actually used in the data generating process.

Another observation is that the confidence set for the case of $p = 0.1$ is included in the confidence set for the case of $p = 0.5$, while the confidence sets for $p = 0.5$ and for $p = 0.9$ are very close to each other. Given that the data generating process for the case of multiple equilibrium outcomes is different (while the realizations of (X_1,X_2) are the same), it is natural to think that the confidence set for the three cases differ from one another. However, we could not analytically show how these are related with each other in this case.

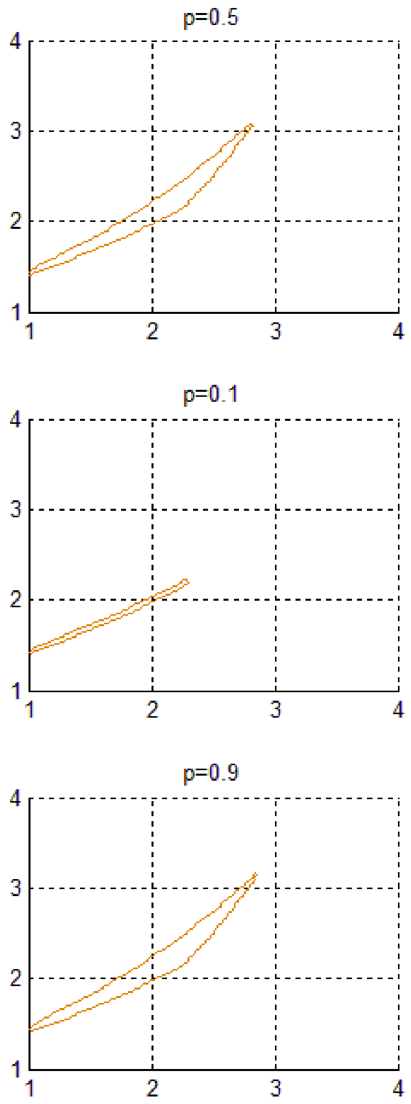

Figure 2: 95% Confidence Set for $p = 0.5$, 0.1, and 0.9. The horizontal axis in θ_1 and the vertical axis is θ_2. True parameter value is $(2, 2)$.

5 Conclusion

This note proposes an approach to estimate supermodular games using moment inequalities. Our approach differs from the approaches taken by the existing studies by addressing the issue of multiplicity of equilibria by adopting a set inference. We also differ form existing studies by using rationalizability as a solution concept, which in general is a weaker restriction than Nash equilibrium. Finally, we conduct Monte Carlo experiments to show that the method works, and presented how the confidence set varies as the data generating process changes.

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