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A note on estimation of two-sided matching models

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ABSTRACT

We propose an estimation strategy for two-sided matching models with non-transferable utility based on the characterization using *pre-matching* that exploits a fixed-point characterization of the set of stable matchings.

1. Introduction

Recently, the number of studies that estimate two-sided matching models has been growing. Although most of the papers consider a two-sided matching model with transferable utility,[1] studies that estimate a two-sided matching model with *non-transferable* utility are scarce. A few exceptions include Boyd et al. (forthcoming), Echenique et al. (forthcoming), Hsieh (2011), and Uetake and Watanabe (2012).

In this note we propose a way to estimate two-sided matching models with non-transferable utility based on the characterization using *pre-matching*, which is proposed by Adachi (2000).[2] This approach provides the characterization of the set of stable matchings as a set of fixed points of the pre-matching mapping. We exploit this fixed-point characterization to estimate the model. While Uetake and Watanabe (2012) use the moment inequality estimator, this paper uses the maximum likelihood. We also propose a computation algorithm.

2. Model

We consider a simple one-to-one two-sided matching model, called the marriage matching problem by Gale and Shapley (1962). In a marriage matching market, there are two types of player: men denoted by $m \in \{1, 2, \ldots, N_m\} \equiv M$ and women denoted by $w \in \{1, 2, \ldots, N_w\} \equiv W$. The utility functions for each m and w, denoted by $U_m(w)$ and $U_w(m)$, are written as follows:

$$U_m(w) = u_m(w) + \varepsilon_{mw},$$
$$U_w(m) = u_w(m) + \varepsilon_{wm},$$

where u_m is an implicit function of observable characteristics of m (denoted by X_m) and of w (denoted by X_w), and ε_{mw} is an idiosyncratic preference shock for m to be matched with w, which an econometrician cannot observe, but players can. Without loss of generality, we can assume that the utility of being single is 0 for all m and w, *i.e.*, $U_m(m) = U_w(w) = 0$. The outcome of the game is a matching μ. A matching $\mu: M \cup W \rightarrow M \cup W$ is a one-to-one correspondence of order two ($\mu(\mu(x)) = x$) such that if $\mu(m) \neq m$, then $\mu(m) \in W$ and if $\mu(w) \neq w$, then $\mu(w) \in M$. For example, if $\mu(m) = w$, then $\mu(w) = m$. This means that m is matched with w in matching μ.

The solution concept we use is the *pairwise stability* defined below.

Definition 1. A matching μ is pairwise stable if the following two conditions are satisfied.

1. (Individual Rationality) $U_m(\mu(m)) \geq U_m(m)$ and $U_w(\mu(w)) \geq U_w(w)$ for all m and w.

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1 Examples, among many, include Choo and Siow (2006), Sørensen (2007), Fox (2010), and Galichon and Salanié (2011).

2 The pre-matching approach is now broadly used in the literature of two-sided matching models including Echenique and Oviedo (2006), Hatfield and Milgrom (2005), and Ostrovsky (2008).

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2. (No-Blocking-Pair Condition) $\nexists(m, w)$ such that $U_m(w) > U_m(\mu(m))$ and $U_w(m) > U_w(\mu(w))$.

Gale and Shapley (1962) prove the existence of pairwise stable matchings using the seminal Deferred Acceptance Algorithm, while Adachi (2000) provides an alternative characterization of the set of stable matchings using the *pre-matching*.

Definition 2. A pair of functions $\upsilon = (\upsilon_M, \upsilon_W)$ is called a pre-matching if $\upsilon_M: M \rightarrow M \cup W$ and $\upsilon_W: W \rightarrow M \cup W$ such that if $\upsilon_M(m) \neq m$, then $\upsilon_M(m) \in W$ and if $\upsilon_W(w) \neq w$, then $\upsilon_W(w) \in M$.

Note that matching μ requires if $\mu(m) = w$, then $\mu(w) = m$, while pre-matching υ does not require such reciprocity. The interpretation of $\upsilon_M(m) = w$ is that man m is willing to be a couple with woman w, but it is not necessarily the case that woman w is also willing to be a couple with man m, *i.e.* $\upsilon_W(w) \neq m$.

Adachi (2000) shows that the set of pairwise stable matchings is the same as the set of solutions of the following equations:

$$\upsilon_M(m) = \arg\max \left\{ u_m(m) + \varepsilon_{m,m}, \max_{w \in W} \left\{ u_m(w) + \varepsilon_{m,w} \middle| \begin{array}{l} u_w(m) + \varepsilon_{w,m} \geq \\ u_w(\upsilon_W(w)) + \varepsilon_{w,\upsilon_W(w)} \end{array} \right\} \right\}, \quad \forall m \in M, \tag{1}$$

$$\upsilon_W(w) = \arg\max \left\{ u_w(w) + \varepsilon_{w,w}, \max_{m \in M} \left\{ u_w(m) + \varepsilon_{w,m} \middle| \begin{array}{l} u_m(w) + \varepsilon_{m,w} \geq \\ u_m(\upsilon_M(m)) + \varepsilon_{m,\upsilon_M(m)} \end{array} \right\} \right\}, \quad \forall w \in W. \tag{2}$$

Pre-matching $\upsilon_M(m)$ specifies the women that man m would like to choose given the pre-matching of all women, υ_W. Note that pre-matching of man m is conditional on the pre-matching of all women, while the pre-matching of all women is conditional on the pre-matching of all men. In a stable matching, $\upsilon_M(m)$ is the best woman w among all women who prefer man m to their current pre-matching $\upsilon_W(w)$. Similarly, $\upsilon_W(w)$ is the best man m among all men who prefer woman w to their current pre-matching $\upsilon_M(m)$.

To present Adachi's (2000) result, let us define the following partial orderings. Define a partial ordering $\geq_M$ on the set of υ_M by $\upsilon_M \geq_M \upsilon'_M$ if and only if $\upsilon_M(m) \succ_m \upsilon'_M(m)\ \forall m$. Similarly, define a partial ordering $\geq_W$ on the set of υ_W by $\upsilon_W \geq_W \upsilon'_W$ if and only if $\upsilon_W(w) \succ_w \upsilon'_W(w)\ \forall w$. Finally, define $\rhd$ on the set of all pre-matchings by $\upsilon \unrhd \upsilon'$ if and only if $\upsilon_M \geq_M \upsilon'_M$ and $\upsilon_W \leq_W \upsilon'_W$.

Proposition 1 (*Adachi, 2000*)**.** *The set of solutions to Eqs. (1) and (2) is non-empty. Hence, the set of stable matchings is non-empty. Furthermore, the set of stable matchings and partial ordering $\unrhd$ form a complete lattice.*

A novel feature of the result of Adachi (2000) is that the set of stable matchings is the set of fixed points of the non-decreasing mapping defined by the right hand side of Eqs. (1) and (2), and that the existence can be proved using Tarski's Fixed Point Theorem.

3. Estimation

Our inference of the model is based on the observations from K independent markets, $k = 1, 2, \ldots, K$. We specify the payoff function as $u_m(w) = u(X_m, X_w, Z_k; \theta)$, where X_m is m's observable characteristics and X_w is w's characteristics, Z_k is market-level characteristics, and θ is the vector of parameters to be estimated.

Let denote the solution (that is stable matching) of Eqs. (1) and (2) by $\upsilon^*_M(m)$ and $\upsilon^*_W(w)$. Then, we denote the probability of $\upsilon^*_M(m)$ being w by $\sigma_m(w)$, and that of $\upsilon^*_W(w)$ being m by $\sigma_w(m)$, *i.e.*, $\sigma_m(w) = \Pr\left(\upsilon^*_M(m) = w\right)$ and $\sigma_w(m) = \Pr\left(\upsilon^*_W(w) = m\right)$. We can interpret $\sigma_m(w)$ to be the choice probability that man m chooses woman w *given pre-matching* σ^*_w. Hence, the probability of obtaining matching between man m and woman w is written as $\sigma_m(w) \times \sigma_w(m)$.

Using $\boldsymbol{\sigma} = (\{\sigma_m\}_{m=1}^{N_m}, \{\sigma_w\}_{w=1}^{N_w})$, and Eqs. (1) and (2), we obtain Eqs. (3) and (4) in the probability space. For any $m \in M$, and for any $w \in W$, $\sigma_m(w)$ and $\sigma_w(m)$ are given in Eqs. (3) and (4) in Box I.

The first part of Eq. (3) is the (conditional) choice probability that w is the optimal choice among all women who prefer m to m', where m' is the current partner of woman w in the pre-matching. In the second part of Eq. (3), $\sigma_w(m')$ is the probability that the current partner of woman w is man m' in the pre-matching. The choice set of man m in the second part of Eq. (3) is all women w' who prefer m to m', *i.e.* $\left\{w' \in W: U_{w'}(m) \geq U_{w'}(m')\right\}$. We define $\widehat{W}_{m,m'} = \left\{w' \in W: U_{w'}(m) \geq U_{w'}(m')\right\}$. Similarly, we define $\widehat{M}_{w,w'} = \left\{m' \in M: U_{m'}(w) \geq U_{m'}(w')\right\}$. Note that these sets are not observed by the econometrician, and they are random sets. If $\{\varepsilon_{m,w}, \varepsilon_{w,m}\}$ follow an *i.i.d.* Type I extreme value distribution, then we can write Eqs. (3) and (4) in the analytical form as follows: for any $m \in M$ and $w \in W$,

$$\sigma_m(w) = \sum_{m' \in M} \sum_{W'} \frac{\exp(u_m(w))}{\sum_{w' \in W' \cup \{m\}} \exp(u_m(w'))} \times \Pr\left(\widehat{W}_{m,m'} = W'\right) \times \sigma_w(m'), \tag{5}$$

$$\sigma_w(m) = \sum_{w' \in W} \sum_{M'} \frac{\exp(u_w(m))}{\sum_{m' \in M' \cup \{w\}} \exp(u_w(m'))} \times \Pr\left(\widehat{M}_{w,w'} = M'\right) \times \sigma_m(w'), \tag{6}$$

where

$$\Pr\left(\widehat{W}_{m,m'} = W'\right) = \prod_{w \in W'} \frac{\exp(u_w(m))}{\exp(u_w(m)) + \exp(u_w(m'))} \times \prod_{w \in W \setminus W'} \frac{\exp(u_w(m'))}{\exp(u_w(m)) + \exp(u_w(m'))},$$

$$\Pr\left(\widehat{M}_{w,w'} = M'\right) = \prod_{m \in M'} \frac{\exp(u_m(w))}{\exp(u_m(w)) + \exp(u_m(w'))} \times \prod_{m \in M \setminus M'} \frac{\exp(u_m(w'))}{\exp(u_m(w)) + \exp(u_m(w'))}.$$

We can compute the probability of choosing unmatched ($\sigma_m(m)$ and $\sigma_w(w)$) by $1 - \sum_{w \in W} \sigma_m(w)$ and $1 - \sum_{m \in M} \sigma_w(m)$, respectively. Note that these equations may correspond to the aggregate quasi-supply and quasi-demand functions in Choo and Siow (2006) though we do not have any transfers and "demand and supply" interpretation may not fit exactly. Moreover, these equations describe the individual-level decisions rather than aggregate demand/supply.

The solution of Eqs. (3) and (4) (or Eqs. (5) and (6)), denoted by $\boldsymbol{\sigma}^* = (\{\sigma^*_m\}_{m \in M}, \{\sigma^*_w\}_{w \in W})$, is the fixed point of the mappings defined by the right hand sides of Eqs. (3) and (4). We can now state the following proposition.

Proposition 2. *The set of fixed points defined by Eqs. (3) and (4) (or Eqs. (5) and (6)) is non-empty.*

Proof. We can apply Brower's Fixed Point Theorem to show the existence of fixed points because $\boldsymbol{\sigma}$ is continuous, and is the mapping from $[0, 1]^{2N}$ onto itself. □

We can solve Eqs. (3) and (4) to get $\boldsymbol{\sigma}^*$ for each market k. Using $\boldsymbol{\sigma}^*$, we can construct a likelihood function. Letting μ_k^{Data} be the observed outcome of market k, the likelihood of observing a match (m, w) in the data can be written as $\sigma^*_m(\mu_k^{Data}(m); \theta) \times \sigma^*_w(\mu_k^{Data}(w); \theta)$. Hence, the likelihood of observing matchings

$$\sigma_m(w) = \sum_{m' \in M} \Pr\left(w = \arg\max \left\{ U_m(m), \begin{array}{l} \max_{w' \in W} \{U_m(w')\} \\ \text{s.t. } U_{w'}(m) \geq U_{w'}(m') \end{array} \right\} \right) \sigma_w(m'), \quad (3)$$

$$\sigma_w(m) = \sum_{w' \in W} \Pr\left(m = \arg\max \left\{ U_w(w), \begin{array}{l} \max_{m' \in M} \{U_w(m')\} \\ \text{s.t. } U_{m'}(w) \geq U_{m'}(w') \end{array} \right\} \right) \sigma_m(w'). \quad (4)$$

Box I.

$\{\mu_k^{Data}\}_{k=1}^K$ is

$$L(\theta) = \prod_{k=1}^{K} \prod_{m=1}^{N_{mk}} \prod_{w=1}^{N_{wk}} \sigma_m^*(\mu_k^{Data}(m); \theta) \times \sigma_w^*(\mu_k^{Data}(w); \theta),$$

where N_{mk} and N_{wk} are the numbers of men and women in market k.

Note that our approach requires the data generating process to correspond to a unique stable matching. An example of obtaining a unique stable matching is the environment in which the data generating process corresponds to men-optimal stable matching. If the econometrician does not have good information about the equilibrium selection mechanism of the data generating process, alternative approaches such as the way proposed by Uetake and Watanabe (2012) can be more helpful.

Finally, we propose a computation procedure. Because the numbers of potential choice sets, $\widehat{W}_{m,m'}$ and $\widehat{M}_{w,w'}$, increase exponentially in the number of players, the exact computation of the mappings in Eqs. (3) and (4) (or (5) and (6)) becomes practically impossible as N_{mk} and N_{wk} increase, and we need some approximation. We propose a computational procedure that approximates the mappings by simulating the choice set.

1. Set the initial choice probabilities in pre-matching, $\boldsymbol{\sigma} = (\{\sigma_m\}_{m \in M}, \{\sigma_w\}_{w \in W})$.
2. Given θ and (m, m'), compute $f_{m,m'}(w) = \Pr(U_w(m) \geq U_w(m'))$ for any $w \in W$.
3. Simulate the choice set, $W^s_{m,m'}$, many times (say, S times) for each (m, m') using $f_{m,m'}$.
4. Compute the conditional choice probability $p^s_{m,m'} = \Pr\Big(w = \arg\max \Big\{ U_m(m), \max_{w' \in W^s_{m,m'}} \{U_m(w')\} \Big\} \Big)$ for each $s = 1, \ldots, S$ and (m, m').
5. Compute $\frac{1}{S} \sum_{s=1}^{S} \sum_{m' \in M} p^s_{m,m'} \times \sigma_m(w')$. Compute the right hand side of Eq. (4) by a similar procedure.
6. Solve Eqs. (3) and (4) until they converge.

Note that $\frac{1}{S} \sum_{s=1}^{S} \sum_{m' \in M} p^s_{m,m'} \times \sigma_m(w')$ converges to the right hand side of Eq. (3) as the number of simulations becomes large.

4. Conclusion

This note proposes an approach to estimate two-sided matching models based on pre-matching. The pre-matching approach provides a fixed point characterization of the set of stable matchings, and it allows us to construct the likelihood function.

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